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Phase and power density distributions on plane apertures of reflector antennas

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Abstract. Complex coordinates are used under the assumptions of geometrical optics to study the relations between phase and power density distributions on plane apertures of reflector antenna systems. No symmetry assumptions are made and the results form a generalisation of existing work in the field. The synthesis of a dual reflector system to produce given aperture phase and power distributions when illuminated by a point source of prescribed power density is shown to depend on the solution of a particular partial differential equation of the Monge–Ampère type. Finally it is shown that even in cases of symmetry the use of complex coordinates simplifies the design equations.

1. Introduction

The majority of papers on the geometrical optics design of reflector antennas assume that the system has some kind of symmetry, for example, rotational symmetry about an axis. The differential equations governing the system are then ordinary differential equations. Brickell *et al* (1977) show that the case of a single reflector with a point source of radiation can be treated in a reasonably simple manner, without such assumptions, by the use of complex coordinates. The incident and reflected ray directions are parametrised by complex coordinates η, ζ respectively and the mapping between them is simply $\zeta=\eta+1/L_{\eta}$, where $L(\eta)$ is a function determining the reflector. The basic differential equations are partial differential equations.

In the present paper we apply similar methods to study the relations between phase and power density distributions on plane apertures of single and dual reflector systems with a point source of radiation. We parametrise source ray direction by the complex coordinate η and points mapped on the aperture plane by the complex coordinate ω. The mapping involves the phase function $l(\omega)$ and can be used either to synthesise the reflector or obtain the power density distribution over the aperture. In the dual system the mapping also depends on the subreflector and it is shown that both phase and power distributions over the aperture of the main reflector are necessary to synthesise the system. We do not make any symmetry assumptions and consequently the paper generalises some of the work of Galindo (1964).

We introduce our notation in § 2. It is first applied in § 3 to study the relation between phase and power density on a plane aperture of a single reflector. The application to dual reflector systems is given in § 4 and in § 5 we specialise to the rotationally symmetric case in order to relate our methods to those of Galindo.

2. Notation and a basic lemma

Unit vectors in space can be parametrised by a complex coordinate in the following way (see figure 1). Choose a rectangular Cartesian coordinate system (x, y, z) and denote its origin by O. Let S denote the unit sphere of centre O. Consider an arbitrary unit vector **OP** where P is some general point on S. Under stereographic projection from the point N(0, 0, 1) P projects to a point P′ in the plane $z = 0$. We given P (and hence the unit vector **OP**) the complex coordinate $\eta = x + \mathrm{i}y$ where $(x, y, 0)$ are the Cartesian coordinates of P′.

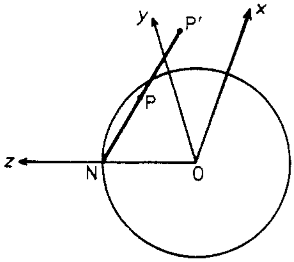

Figure 1. Diagram showing coordinate system.

A general vector $\boldsymbol{a}$ has a set of components (a_1, a_2, a_3) with respect to the Cartesian coordinate system. We shall write (α, a_3) for this set where α is the complex number $a_1 + \mathrm{i}a_2$. With this notation the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$ is given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = \tfrac{1}{2}(\alpha\bar{\beta} + \bar{\alpha}\beta) + a_3 b_3$$

where $\beta = b_1 + \mathrm{i}b_2$ and a bar denotes the complex conjugate.

For example, a calculation shows that the components of the unit vector **OP** are expressed in terms of its complex coordinate η as

$$\left(\frac{2\eta}{|\eta|^2+1}, \frac{|\eta|^2-1}{|\eta|^2+1}\right) \tag{1}$$

where $|\eta|$ is the modulus of η. It follows that the standard spherical polar coordinates of P are related to η by

$$\mathrm{e}^{\mathrm{i}\phi} \sin\theta = \frac{2\eta}{|\eta|^2+1}, \qquad \cos\theta = \frac{|\eta|^2-1}{|\eta|^2+1}. \tag{2}$$

We shall also need an expression for the complex coordinate ν of the unit vector in the direction of the non-zero vector $\boldsymbol{a} = (\alpha, a_3)$. A short calculation shows that

$$\nu = \alpha/(a - a_3) \tag{3}$$

where a is the length of $\boldsymbol{a}$.

A technical lemma which will be used later is now stated and proved.

Lemma 1. Let $\boldsymbol{b}$ be a vector of components (β, g), $\boldsymbol{k}$ a unit vector of complex coordinate μ and m a scalar. Define real scalars H, K by $H = m^2 - b^2$, $K = m - \boldsymbol{k} \cdot \boldsymbol{b}$. Then, provided that $K > 0$, the equation

$$\boldsymbol{a} - a\boldsymbol{k} = \boldsymbol{b} - m\boldsymbol{k} \tag{4}$$

has the unique solution

$$\boldsymbol{a} = \boldsymbol{b} - (H/2K)\boldsymbol{k}. \tag{5}$$

The length a of $\boldsymbol{a}$ is given by

$$a = m - (H/2K), \tag{6}$$

and its direction by the unit vector of complex coordinate

$$\nu = (g + m - \beta\bar{\mu})/(\bar{\beta} + (g - m)\bar{\mu}). \tag{7}$$

Proof. Any solution to equation (4) is necessarily of the form $\boldsymbol{b} + \lambda\boldsymbol{k}$ where λ is a scalar and this expression is a solution if, and only if, there exists $a > 0$ such that

$$\lambda + m = a, \qquad a^2 = b^2 + \lambda^2 + 2\lambda \boldsymbol{b} \cdot \boldsymbol{k}.$$

Substituting for a in the second condition we find that $\lambda = -H/2K$. Consequently any solution is unique. The fact that

$$\boldsymbol{a} = \boldsymbol{b} - (H/2K)\boldsymbol{k}$$

is a solution follows from the inequality

$$a = m - (H/2K) = |\boldsymbol{b} - m\boldsymbol{k}|^2/2K > 0$$

where $|\boldsymbol{b} - m\boldsymbol{k}|$ denotes the length of the vector $\boldsymbol{b} - m\boldsymbol{k}$. This remark also justifies the formula (6).

To prove the formula (7) we use the equation (3) to obtain

$$\nu = \frac{(|\mu|^2 + 1)K\beta - H\mu}{(|\mu|^2 + 1)K(m - g) - H}.$$

The formulae

$$(|\mu|^2 + 1)K = (m - g)|\mu|^2 + m + g - \mu\bar{\beta} - \bar{\mu}\beta$$
$$H = m^2 - g^2 - \beta\bar{\beta}$$

can be obtained from the definitions of H and K. We substitute these in the expression for ν to obtain

$$\nu = \frac{\beta(m - g)|\mu|^2 + \beta(m + g) - \bar{\mu}\beta^2 + (g - m)(g + m)\mu}{(g - m)^2|\mu|^2 + (g - m)(\mu\bar{\beta} + \bar{\mu}\beta) + \beta\bar{\beta}}$$
$$= \frac{(g + m - \beta\bar{\mu})(\beta + (g - m)\mu)}{(\bar{\beta} + (g - m)\bar{\mu})(\beta + (g - m)\mu]} = \frac{g + m - \beta\bar{\mu}}{\bar{\beta} + (g - m)\bar{\mu}}.$$

Thus the lemma is proved.

Because we are using a complex coordinate it is convenient to introduce derivatives with respect to this coordinate. Given a function f of $\eta = x + iy$ (in general, f is complex-valued) we define

$$f_\eta = \frac{\partial f}{\partial \eta} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad f_{\bar{\eta}} = \frac{\partial f}{\partial \bar{\eta}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

These derivatives commute, that is $f_{\eta\bar\eta}=f_{\bar\eta\eta}$ and also satisfy the relations

$$\bar f_\eta=\overline{(f_{\bar\eta})}, \qquad \bar f_{\bar\eta}=\overline{(f_\eta)}$$

where $\bar f$ is the complex conjugate function.

The partial differential equations which arise in this paper have the form

$$(L_{\eta\bar\eta}-b)^2-|L_{\eta\eta}+a|^2=d \tag{8}$$

where L is a real-valued function of η and the coefficients a, b, d are functions of η, L and first-order partial derivatives of L. In addition the functions b, d are real valued. It can be shown that the equation is a Monge–Ampère differential equation and that it is of elliptic (hyperbolic) type if $d>0$ ($d<0$).

3. Phase and power density on a plane aperture of a single reflector

Figure 2 shows the path of a ray from the source O of radiation. It is reflected at the point R on the reflector and passes through the plane aperture at Q from left to right. Unit vectors in the directions of the rays at O and Q are denoted by $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively. We write $\mathbf{OR}=\boldsymbol{r}$ and $\mathbf{OQ}=\boldsymbol{v}$ and choose a rectangular Cartesian coordinate system with origin at O and z axis perpendicular to the aperture. The complex coordinates of $\boldsymbol{p}$, $\boldsymbol{q}$ relative to this system are denoted by η, ζ respectively. The components of $\boldsymbol{v}$ are expressed as (ω, d). Thus d is the perpendicular distance of O from the aperture and we can regard ω as a complex coordinate for the point Q on the aperture.

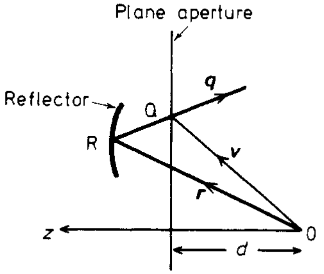

Figure 2. Ray diagram for single reflector.

We assume that there is a unique ray at each point of the aperture so that a phase function $l(\omega)$ is defined. This prohibits any caustic points *on* the aperture. The function $l(\omega)$ determines ζ as a function ω. To see this we first apply the theorem of Malus (see, for example, Cornbleet 1976, p 361) which shows that $\boldsymbol{q}\,.\,\mathrm{d}\boldsymbol{v}=\mathrm{d}l$ that is

$$\frac{1}{1+|\zeta|^2}(\bar\zeta\,\mathrm{d}\omega+\zeta\,\mathrm{d}\bar\omega)=dl.$$

It follows that

$$l_\omega=\bar\zeta/(1+|\zeta|^2), \qquad l_{\bar\omega}=\zeta/(1+|\zeta|^2). \tag{9}$$

We proceed to solve these equations for ζ. We have at once that

$$|l_\omega|=|\zeta|/(1+|\zeta|^2)$$

which is a quadratic for $|\zeta|$. Since $\boldsymbol{q}$ points to the right of the aperture we must choose the root which is less than or equal to unity.

Consequently

$$|\zeta| = (1-(1-4|l_\omega|^2)^{1/2})/2|l_\omega|$$

and therefore

$$\zeta = 2l_{\bar{\omega}}/(1+(1-4|l_\omega|^2)^{1/2}). \tag{10}$$

We note that the phase function necessarily satisfies the inequality $|l_\omega| \leqslant \frac{1}{2}$.

The formulae

$$\zeta_\omega = \left(\frac{1+|\zeta|^2}{1-|\zeta|^2}\right)^2 (\zeta^2 l_{\omega\omega} + l_{\omega\bar{\omega}}), \qquad \zeta_{\bar{\omega}} = \left(\frac{1+|\zeta|^2}{1-|\zeta|^2}\right)(\zeta^2 l_{\omega\bar{\omega}} + l_{\bar{\omega}\bar{\omega}}) \tag{11}$$

can be obtained from equation (10). They will be used later.

Our aim in this section is to establish a relation between phase and power density over the aperture. The equation

$$\boldsymbol{r} - r\boldsymbol{q} = \boldsymbol{v} - l\boldsymbol{q}$$

is easily obtained and lemma 1 shows that $\boldsymbol{r}$ is given by

$$\boldsymbol{r} = \boldsymbol{v} - (H/2K)\boldsymbol{q} \tag{12}$$

where

$$H = l^2 - |\omega|^2 - d^2 \tag{13}$$

$$(1+|\zeta|^2)K = l(1+|\zeta|^2) + d(1-|\zeta|^2) - \zeta\bar{\omega} - \bar{\zeta}\omega. \tag{14}$$

The coordinate ζ is given in terms of ω by equation (10) and so the reflector, given by equation (12), is determined in terms of ω by the phase function.

Lemma 1 also shows that the complex coordinate η of the unit vector $\boldsymbol{p}$ is given by

$$\eta = \frac{l+d-\omega\bar{\zeta}}{\bar{\omega}+(d-l)\bar{\zeta}}. \tag{15}$$

We regard this equation as determining a mapping $\omega \to \eta$ between the aperture plane and the unit sphere of centre O. It will alter areas (see appendix) by the factor

$$\frac{4}{(1+|\eta|^2)^2}\left||\eta_\omega|^2 - |\eta_{\bar{\omega}}|^2\right|.$$

Therefore, if $I(\eta)$ denotes the power density of the source and $G(\omega)$ the power density flow normal to the aperture

$$\frac{G(\omega)}{I(\eta)} = \frac{4}{(1+|\eta|^2)^2}\left||\eta_\omega|^2 - |\eta_{\bar{\omega}}|^2\right|. \tag{16}$$

Once we have calculated the partial derivatives η_ω, $\eta_{\bar{\omega}}$ equation (16) leads to the desired relation between l and G.

Straightforward calculations using equations (15) and (9) lead to the formulae

$$\eta_\omega = \frac{H\bar{\zeta}_\omega + \bar{\zeta}^2 K}{(\bar{\omega}+(d-l)\bar{\zeta})^2}, \qquad \eta_{\bar{\omega}} = \frac{H\bar{\zeta}_{\bar{\omega}} - K}{(\bar{\omega}+(d-l)\bar{\zeta})^2}. \tag{17}$$

We may also use equation (15) to show that

$$(1+|\eta|^2)|\bar{\omega}+(d-l)\bar{\zeta}|^2=(1+|\zeta|^2)(2lK-H). \tag{18}$$

Finally, by substituting these results in (16) and using equations (9) and (11) to replace ζ and its derivatives by those of l, we obtain

$$[hl_{\omega\bar{\omega}}-(1-2|l_\omega|^2)]^2-|hl_{\omega\omega}+2l_{\bar{\omega}}^2|^2=\mp(l-\tfrac{1}{2}h)^2(1-4|l_\omega|^2)^{1/2}\frac{G(\omega)}{I(\eta)} \tag{19}$$

where $h=H/K$. We note that from equations (13), (14) and (15) h and η can be expressed in terms of l and its first derivatives.

One can regard the relation (19) in two ways. If the density function $G(\omega)$ is specified on the aperture then it becomes (as can be seen from equation (8)) a Monge–Ampère differential equation for l. The reflector is uniquely determined once a solution is known.

On the other hand if the phase function $l(\omega)$ is specified then equation (19) is a formula for the power density function $G(\omega)$. For example, suppose that $l(\omega)$ is required to be constant along all lines in the aperture plane parallel to a fixed direction (this condition characterises the reflector designs of Dunbar 1948). We take this direction as the direction of the y axis so that l is a function f of x only.

We find

$$l_\omega=l_{\bar{\omega}}=\tfrac{1}{2}f', \qquad l_{\omega\omega}=l_{\omega\bar{\omega}}=\tfrac{1}{4}f''$$

and equation (19) gives

$$\frac{G}{I}=\frac{\mp 2[2-2(f')^2-hf'']}{(2f-h)^2[1-(f')^2]^{1/2}} \tag{20}$$

where $h=H/K=(f^2-x^2-y^2-d^2)/\{f-xf'+d[1-(f')^2]^{1/2}\}$. The function f can be chosen to give a specified form for G/I over the central line $y=0$ in the aperture.

In particular we can take f to be a constant. Then with $A=f+d$, equation (20) becomes

$$\frac{G}{I}=\frac{4A^2}{(A^2+x^2+y^2)^2},$$

a well known formula for the parabolic reflector (see for example Collin and Zucker 1969).

4. Phase and power density on a plane aperture of a dual reflector system

Figure 3 shows the path of a ray from the source O of radiation. It is reflected at the points R, S on the reflectors and passes through the plane aperture of the second reflector at Q from left to right. We denote the unit vectors in the directions of the ray at O, R and Q by $\boldsymbol{p}$, $\boldsymbol{t}$ and $\boldsymbol{q}$ respectively. We write $\mathbf{OR}=\boldsymbol{r}$, $\mathbf{RS}=\boldsymbol{s}$ and $\mathbf{OQ}=\boldsymbol{v}$. We also choose a rectangular Cartesian coordinate system with origin at O and z axis perpendicular to the aperture. The complex coordinates of $\boldsymbol{p}$, $\boldsymbol{t}$, $\boldsymbol{q}$ relative to this system are noted by η, ξ, ζ respectively. As in § 3 we express the components of $\boldsymbol{v}$ as (ω, d) so that d is again the perpendicular distance of O from the aperture and ω is a complex coordinate for Q.

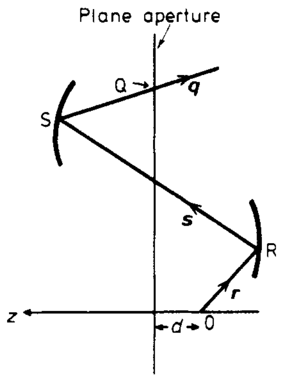

Figure 3. Ray diagram for dual reflector system.

We suppose that there are no caustic points on the aperture so that a phase function $l(\omega)$ is defined. As in § 3 $l(\omega)$ determines ζ as a function of ω, the explicit form being given in equation (10).

Our aim in this section is again to establish a relation between phase and power density over the aperture. In the dual reflector system this relation will involve the first reflector. Therefore there is the possibility of designing the first reflector to give pre-assigned phase and power density distributions over the aperture.

Let us suppose the first reflector to be given by an equation $\boldsymbol{r} = \boldsymbol{r}(\eta)$. The Cartesian components of $\boldsymbol{r}$ are

$$r\left(\frac{2\eta}{1+|\eta|^2}, \frac{|\eta|^2-1}{|\eta|^2+1}\right).$$

It is convenient to introduce the function $\tau = r/(1+|\eta|^2)$ so that the above components can be written as

$$(2\eta\tau, \tau(|\eta|^2-1)). \tag{21}$$

It is also convenient to introduce the function $L = \ln \tau$. Brickell *et al* (1977) show that Snell's law at the first reflection can be expressed in terms of this function as

$$L_\eta = 1/(\xi - \eta). \tag{22}$$

The relation

$$\boldsymbol{s} - s\boldsymbol{q} = \boldsymbol{v} - \boldsymbol{r} - (l - r)\boldsymbol{q}$$

is easily established. It follows from equation (21) that

$$\boldsymbol{v} - \boldsymbol{r} = (\omega - 2\eta\tau, d - \tau(|\eta|^2 - 1))$$

and consequently, from formula (7) of lemma 1,

$$\xi = \frac{l + d - 2\tau|\eta|^2 - (\omega - 2\eta\tau)\bar{\zeta}}{\bar{\omega} - 2\bar{\eta}\tau + (2\tau + d - l)\bar{\zeta}}.$$

We substitute this expression for ξ into equation (22) and obtain

$$L_\eta = \frac{\bar{\omega} - 2\bar{\eta}\tau + (2\tau + d - l)\bar{\zeta}}{l + d - \omega\bar{\zeta} - \bar{\omega}\eta + (l - d)\bar{\zeta}\eta}. \tag{23}$$

We shall regard equation (23) as establishing a mapping $\eta \to \omega$ between the unit sphere of centre O and the aperture. Of course this mapping depends on the phase function and the first reflector. The formula (5) of lemma 1 shows that

$$\boldsymbol{r}+\boldsymbol{s}=\boldsymbol{v}-(H/2K)\boldsymbol{q}, \tag{24}$$

where H and K are functions of ω and η which we shall shortly make explicit. Thus the second reflector is determined in terms of η by the first reflector and the phase function.

Explicit formulae for H and K are as follows:

$$\begin{aligned} H &= (l-r)^2-|\boldsymbol{v}-\boldsymbol{r}|^2 = l^2-|\boldsymbol{v}|^2-2r(l-\boldsymbol{p}\cdot\boldsymbol{v}) \\ &= l^2-|\omega|^2-d^2-2\tau(l(1+|\eta|^2)+d(1-|\eta|^2)-\bar{\eta}\omega-\eta\bar{\omega}). \end{aligned} \tag{25}$$

$$\begin{aligned} K &= (l-r)-\boldsymbol{q}\cdot(\boldsymbol{v}-\boldsymbol{r}) \\ &= [l(1+|\zeta|^2)+d(1-|\zeta|^2)-\bar{\zeta}\omega-\zeta\bar{\omega}+2\tau(\bar{\eta}\zeta+\eta\bar{\zeta}-|\eta|^2-|\zeta|^2)]/(1+|\zeta|^2). \end{aligned} \tag{26}$$

Let $I(\eta)$ denote the source power density and $G(\omega)$ the power density flow normal to the aperture. Then, by the same argument as that leading to equation (16),

$$\frac{I(\eta)}{G(\omega)}=\frac{(1+|\eta|^2)^2}{4}\left||\omega_\eta|^2-|\omega_{\bar{\eta}}|^2\right|. \tag{27}$$

Once we have calculated the partial derivatives ω_η, $\omega_{\bar{\eta}}$ from equation (23) then equation (27) provides the desired relation between the functions l, G and L.

To do this calculation it is convenient to think of the right-hand side of equation (23) as a function $F(\omega, \eta)$ of the variables ω, η. This is achieved by regarding l, ζ as functions of ω, and τ as a function of η. We find the partial derivatives of F to be

$$F_\omega=(-H\bar{\zeta}_\omega-K\bar{\zeta}^2)/R^2, \qquad F_{\bar{\omega}}=(-H\bar{\zeta}_{\bar{\omega}}+K)/R^2 \tag{28}$$

$$F_\eta=L_\eta^2, \qquad F_{\bar{\eta}}=-2\tau K(1+|\zeta|^2)/|R|^2 \tag{29}$$

where

$$R=l+d-\omega\bar{\zeta}-\bar{\omega}\eta+(l-d)\bar{\zeta}\eta.$$

In the last pair of these formulae equation (23) has been used. This equation is also used to give the formula

$$|R|^2=\frac{(l+d-2\tau|\eta|^2)(1+|\zeta|^2)K-|\zeta|^2H}{|1+\eta L_\eta|^2}. \tag{30}$$

We differentiate the equation

$$L_\eta=F(\omega, \eta)$$

and obtain

$$L_{\eta\eta}=F_\omega\omega_\eta+F_{\bar{\omega}}\bar{\omega}_\eta+L_\eta^2,$$

$$L_{\eta\bar{\eta}}=F_\omega\omega_{\bar{\eta}}+F_{\bar{\omega}}\bar{\omega}_{\bar{\eta}}+b,$$

where $b=F_{\bar{\eta}}$. We note, from equation (29), that b is real. These equations and their complex conjugates can be combined in the simple matrix equation

$$\begin{bmatrix} L_{\eta\eta}-L_\eta^2 & L_{\eta\bar{\eta}}-b \\ L_{\eta\bar{\eta}}-b & L_{\bar{\eta}\bar{\eta}}-L_{\bar{\eta}}^2 \end{bmatrix}=\begin{bmatrix} F_\omega & F_{\bar{\omega}} \\ \bar{F}_\omega & \bar{F}_{\bar{\omega}} \end{bmatrix}\begin{bmatrix} \omega_\eta & \omega_{\bar{\eta}} \\ \bar{\omega}_\eta & \bar{\omega}_{\bar{\eta}} \end{bmatrix}.$$

Taking the determinants of both sides we obtain

$$|L_{\eta\eta} - L_{\eta}^2|^2 - (L_{\eta\bar{\eta}} - b)^2 = (|F_{\omega}|^2 - |F_{\bar{\omega}}|^2)(|\omega_{\eta}|^2 - |\omega_{\bar{\eta}}|^2).$$

This is essentially the relationship we require. However we can make it more explicit by using the formulae (9), (11), (27), (28), (29) and (30). With the notation

$$\beta = l + d - 2\,\mathrm{e}^L|\eta|^2, \qquad \gamma = |1 + \eta L_{\eta}|^2, \qquad \delta = 1 - (1 - 4|l_{\omega}|^2)^{1/2}, \qquad h = H/K$$

we find that

$$(L_{\eta\bar{\eta}} - b)^2 - |L_{\eta\eta} - L_{\eta}^2|^2 = \pm B\{[hl_{\omega\bar{\omega}} - (1 - 2|l_{\omega}|^2)]^2 - |hl_{\omega\omega} + 2l_{\omega}^2|^2\} I(\eta)/G(\omega) \tag{31}$$

where

$$b = \frac{-2\mathrm{e}^L\gamma}{(\beta - \frac{1}{2}h\delta)}, \qquad \text{and} \qquad B = \frac{4\gamma^2}{(1-\delta)(\beta - \frac{1}{2}h\delta)^2(1 + |\eta|^2)^2}.$$

The relation (31) can be regarded in many ways. If the functions $l(\omega)$, $G(\omega)$ are given we can regard it as a partial differential equation for L, provided that we substitute for ω the function of η, L and its first derivatives obtained by solving equation (23). For example suppose that l is a constant function on the aperture. Equation (31) becomes

$$(L_{\eta\bar{\eta}} - b)^2 - |L_{\eta\eta} - L_{\eta}^2|^2 = \pm BI(\eta)/G(\omega)$$

where

$$b = -2\,\mathrm{e}^L\gamma/\beta, \qquad B = 4\gamma^2/\beta^2(1 + |\eta|^2)^2.$$

In this special case $\zeta = 0$ (from equation (10)) and the solution of equation (23) is

$$\omega = \frac{(l + d)L_{\bar{\eta}} + 2\eta\,\mathrm{e}^L}{1 + \bar{\eta}L_{\bar{\eta}}}. \tag{32}$$

We work out the relationship (32) explicitly for the well known hyperboloid–paraboloid Cassegrain system (see, for example, Cornbleet 1976, p 85). In this system the first reflector (the subreflector) is a hyperboloid of revolution and the source is at one focus. The second reflector (the main reflector) is a paraboloid of revolution with focus at the second focus of the hyperboloid and axis in the direction $\zeta = 0$. The configuration assumed is shown in figure 4 where the hyperboloid axis is tilted relative to the paraboloid axis. In this system the shadowing by the subreflector of the main reflector aperture can be minimised.

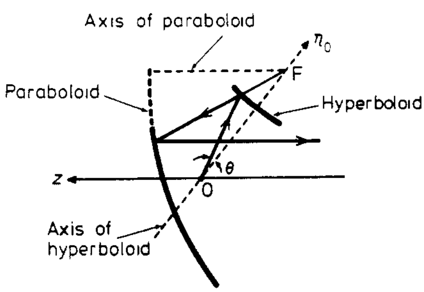

Figure 4. 'Open' Cassegrain system.

Let the direction of the hyperboloid axis be η_0. Then the equation of the hyperboloid is

$$a/r = e\cos\theta - 1,$$

where a is the semi-latus rectum and e the eccentricity. Now

$$\cos\theta = \frac{2(\eta\bar{\eta}_0 + \bar{\eta}\eta_0) + (|\eta|^2 - 1)(|\eta_0|^2 - 1)}{(1+|\eta_0|^2)(1+|\eta|^2)}$$

and so if we write

$$\alpha = \frac{2\eta_0}{1+|\eta_0|^2}, \qquad k = \frac{|\eta_0|^2 - 1}{|\eta_0|^2 + 1}$$

we obtain

$$e^L = \frac{r}{1+|\eta|^2} = \frac{a}{(1+|\eta|^2)(e\cos\theta - 1)} = \frac{a}{e\bar{\alpha}\eta + e\alpha\bar{\eta} + |\eta|^2(ek-1) - (1+ek)}.$$

It follows that

$$L_{\bar{\eta}} = -\frac{e\alpha + \eta(ek-1)}{e\bar{\alpha}\eta + e\alpha\bar{\eta} + |\eta|^2(ek-1) - (1+ek)}.$$

Thus putting $A = l + d$, the relation between η and ω turns out to be the bilinear one

$$\omega = \frac{[2a - A(ek-1)]\eta - Ae\alpha}{e\bar{\alpha}\eta - (1+ek)}.$$

We use the formula to obtain an explicit expression for $G(\omega)/I(\eta)$ from equation (16). We first invert it to get

$$\eta = \frac{(1+ek)\omega - Ae\alpha}{e\bar{\alpha}\omega + [A(ek-1) - 2a]}$$

and then we differentiate, noting that $k^2 + |\alpha|^2 = 1$, to obtain

$$\eta_\omega = \frac{A(e^2-1) - 2a(1+ek)}{\{e\bar{\alpha}\omega + [A(ek-1) - 2a]\}^2}, \qquad \eta_{\bar{\omega}} = 0.$$

We substitute from these formulae into equation (16) and find that

$$\frac{G(\omega)}{I(\eta)} = \frac{4[A(e^2-1) - 2a(ek+1)]^2}{[|e\bar{\alpha}\omega + A(ek-1) - 2a|^2 + |(ek+1)\omega - Ae\alpha|^2]^2}.$$

5. Dual reflector systems with rotational symmetry

The coordinates are assumed to be set up as in § 4 and we suppose also that the source has a power density I which has rotational symmetry about the z axis. We now consider the problem of designing a dual reflector system which has the z axis as an axis of symmetry and which produces a specified phase function l and power density function G on the aperture, where l, G are also rotationally symmetric about the z axis.

We put $|\eta| = X$, $|\omega| = Y$. The symmetry assumptions imply that I is a function of X and l, G are functions of Y. To calculate a derivative, say $l_{\bar{\omega}}$, we proceed as follows. Since $Y^2 = \omega\bar{\omega}$ we have $2YY_{\bar{\omega}} = \omega$ and consequently

$$l_{\bar{\omega}} = l'(Y)Y_{\bar{\omega}} = l'(Y)\omega/2Y.$$

A major simplification in the rotationally symmetric case is that equation (27) produces at once a differential equation for Y as a function of X. To obtain this equation we note that for rotationally symmetric systems

$$\omega = \pm Y_\eta/X. \tag{33}$$

Let Y' denote the derivative of Y with respect to X. Then it follows from the relation (33) that

$$\omega_\eta = \pm\frac{1}{2}\left(\frac{Y}{X} + Y'\right), \qquad \omega_{\bar{\eta}} = \pm\frac{1}{2}\left(\frac{\eta}{\bar{\eta}}\right)\left(Y' - \frac{Y}{X}\right)$$

and therefore

$$|\omega_\eta|^2 - |\omega_{\bar{\eta}}|^2 = YY'/X.$$

Thus equation (27) leads to one or other of the ordinary differential equations

$$YY'G(Y) = \pm\frac{4X}{(1+X^2)^2}I(X) \tag{34}$$

where we note that the choices of sign in equations (33) and (34) are not related.

We can determine the first reflector using equations (10), (23) and (34). It follows from equation (10) that

$$\zeta = \omega l'(Y)/YZ$$

where we have written Z for the expression $1 + [1 - (l'(Y))^2]^{1/2}$. For a rotationally symmetric reflector the function τ will be a function of X only. Consequently, substituting for ζ in equation (23) and making use of the relation (33), we obtain

$$\frac{\tau'(X)}{2\tau} = \frac{(\pm Y - 2X\tau)Z \mp (l - d - 2\tau)l'(Y)}{(l + d \mp XY)Z + [\pm(l-d)X - Y]l'(Y)}. \tag{35}$$

The choice of sign in this equation is related to the choice in the relation (33). Thus, having solved the differential equation (34) to obtain Y as a function of X, we can substitute this function into equation (35). The result is a choice of two ordinary differential equations for τ which can be solved subject to an initial condition.

The final step is the determination of the second reflector which is easily obtained using equations (24), (25) and (26).

As an example of the design of rotationally symmetric systems we follow Galindo (1964) and assume that

$$I(X) = \left(\frac{1-X^2}{1+X^2}\right)^n, \qquad n \geqslant 0$$

and suppose that both l and G are constant functions on the plane aperture.

The differential equations (34) become

$$YY'G = \pm\frac{4X(1-X^2)^n}{(1+X^2)^{n+2}}$$

which integrates to give

$$\tfrac{1}{2}GY^2 = \mp\frac{1}{(n+1)}\left(\frac{1-X^2}{1+X^2}\right)^{n+1} + C,$$

where C is a constant of integration. We will design the system to include a ray along the z axis so that $Y(0)=0$. Consequently $C=\pm 1/(n+1)$. It is convenient to write $G=2M/(n+1)$ where M is a positive constant. Then we obtain

$$MY^2 = 1-\left(\frac{1-X^2}{1+X^2}\right)^{n+1} \tag{36}$$

as only the choice of negative sign is sensible. It follows that

$$Y=\left\{\left[1-\left(\frac{1-X^2}{1+X^2}\right)^{n+1}\right]M^{-1}\right\}^{1/2}.$$

The equation (35) becomes, in the present special case

$$\frac{\mathrm{d}\tau}{\mathrm{d}X}=\frac{2\tau(\pm Y-2X\tau)}{A\mp XY}$$

where A is the constant $l+d$ and Y is the function of X obtained in equation (36). Given an initial value $\tau(0)$ $(=r(0))$ we can solve either of the resulting differential equations and so obtain a first reflector. Indeed the substitution $\sigma=1/\tau$ produces the *linear* differential equations

$$\frac{\mathrm{d}\sigma}{\mathrm{d}X}\pm\left(\frac{2Y}{A\mp XY}\right)\sigma=\frac{4X}{A\mp XY} \tag{37}$$

which can be solved explicitly in terms of integrations. The resulting reflector profiles are given by transcendental functions of X rather than conic sections as in the Cassegrain systems.

The case in which $n=1$ corresponds to the Schwarzchild system which has some useful properties enabling it to be considered by White and DeSize (1962) as a prototype for a scanning antenna system. For this case the solution of equation (37) yields

$$\sigma=\frac{2X^2}{A}+K[A+(A\mp F)X^2]^{\mp F/(A\mp F)} \tag{38}$$

where $F=2M^{-1/2}$ and K is a constant determined by $\sigma(0)$.

The second reflector may be obtained by using equation (24) to obtain the components of $\boldsymbol{r}+\boldsymbol{s}$ as

$$\left[\pm\frac{Y\eta}{X},\frac{1}{2}A-\frac{[\sigma Y^2+2(A\mp 2XY)]}{2(\sigma A-2X^2)}\right]$$

where σ is given by (38) and $Y=FX/(1+X^2)$.

Appendix

Let $\omega\to\eta(\omega)$ be a mapping between the aperture and the unit sphere of centre O expressed in terms of the complex coordinates η, ω. We wish to show that the mapping alters areas by the factor

$$\frac{4}{(1+|\eta|^2)^2}\left||\eta_\omega|^2-|\eta_{\bar{\omega}}|^2\right|. \tag{A.1}$$

In spherical polar coordinates (θ, ϕ) the area element on the unit sphere is $\sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$. Thus, if the mapping is expressed in terms of (θ, ϕ) and rectangular Cartesian coordinates (x, y) on the aperture, the formula for the factor is

$$\sin\theta\left|\frac{\partial(\theta, \phi)}{\partial(x, y)}\right|, \tag{A.2}$$

where $\partial(\theta, \phi)/\partial(x, y)$ is the Jacobian determinant

$$\begin{vmatrix} \theta_x & \theta_y \\ \phi_x & \phi_y \end{vmatrix}.$$

We shall use the coordinates (θ, ϕ) given in terms of η by the equations (2) and the coordinates (x, y) given by $\omega = x + \mathrm{i}y$.

A well known property of Jacobian determinants enables us to write

$$\frac{\partial(\theta, \phi)}{\partial(x, y)} = \frac{\partial(\theta, \phi)}{\partial(\eta, \bar{\eta})}\frac{\partial(\eta, \bar{\eta})}{\partial(\omega, \bar{\omega})}\frac{\partial(\omega, \bar{\omega})}{\partial(x, y)} \tag{A.3}$$

and we shall deduce the formula (A.1) from (A.2) by evaluating the determinants on the right-hand side. We obtain from the equations (2) that

$$\cos\theta = 1 - \frac{2}{1+|\eta|^2}, \qquad \phi = \frac{1}{2\mathrm{i}}\ln(\eta/\bar{\eta})$$

and consequently

$$\sin\theta\,\theta_\eta = \frac{-2\bar{\eta}}{(1+|\eta|^2)^2}, \qquad \phi_\eta = \frac{1}{2\mathrm{i}\eta}.$$

It follows that

$$\sin\theta\frac{\partial(\theta, \phi)}{\partial(\eta, \bar{\eta})} = \frac{2}{\mathrm{i}}\frac{1}{(1+|\eta|^2)^2}. \tag{A.4}$$

Because

$$\frac{\partial(\omega, \bar{\omega})}{\partial(x, y)} = -2\mathrm{i}, \qquad \frac{\partial(\eta, \bar{\eta})}{\partial(\omega, \bar{\omega})} = |\eta_\omega|^2 - |\eta_{\bar{\omega}}|^2$$

formula (A.1) is a consequence of (A.2) and the equations (A.3) and (A.4).

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